6.3 - Solutions About Singular Points

Due Sat

A singular point for the equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ (*) is a point $x = x_0$ such that $a_2(x_0) = 0$. For our work with solving differential equations, we will only consider $x_0 = 0$. Standard form for (*) is $y'' + P(x)y' + Q(x)y = 0$. Multiplying by x^2 (or by $(x - x_0)^2$ if $x_0 \neq 0$) yields
$x^2y'' + x^2P(x)y' + x^2Q(x)y = 0$. By relabeling $p(x) = xP(x)$, $q = x^2Q(x)$, we get $x^2y'' + xp(x)y' + q(x)y = 0$.

Definition: $x = 0$ is a **regular singular point** if $p(x)$ and $q(x)$ are analytic at $x = 0$ and an **irregular singular point** if either $p(x)$ or $q(x)$ is not analytic at $x = 0$.

Example: Determine the singular points of the given differential equation. Classify each singular point as regular or irregular.

$x(x + 3)^2y'' - y = 0$

$x=0$ & $x=-3$ are singular points

Std form: $y'' - \frac{1}{x(x+3)^2}y = 0$

$x=0$: $x^2y'' - \frac{x}{(x+3)^2}y = 0$

$x=-3$ $(x+3)^2y'' - \frac{1}{x}y = 0$

$x=0$ is regular

$x=-3$ is regular

Example: Determine the singular points of the given differential equation. Classify each singular point as regular or irregular.

$$y'' - \frac{1}{x}y' + \frac{1}{(x-1)^3}y = 0$$

$x = 0$, $x = 1$ are singular

$x = 0$: $x^2 y'' - x y' + \frac{x^2}{(x-1)^3} y = 0$ regular

$x = 1$: $(x-1)^2 y'' - \frac{(x-1)^2}{x} y' + \frac{1}{x-1} y = 0$ irregular

Theorem: (Frobenius' Theorem) If $x = x_0$ is a regular singular point of (∗), then there is at least one solution of the form $y = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$, where the number r is a constant to be determined.

Example: $x = 0$ is a regular singular point of the given differential equation. Show that the indicial roots of the singularity do not differ by an integer. Use the method of Frobenius to obtain two linearly independent solutions about $x = 0$. Form the general solution on $(0, \infty)$.

$$2xy'' + 5y' + xy = 0$$

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$$

Note: the index does not increase here

Clairvoyence: r will turn out to be $r = 0, -\frac{3}{2}$

$$y = c_0 x^{-3/2} + c_1 x^{-1/2} + c_2 x^{1/2} + \cdots$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) c_n x^{\underline{n+r-1}} + \sum_{n=0}^{\infty} 5(n+r) c_n x^{\underline{n+r-1}}$$

$$+ \sum_{n=0}^{\infty} c_n x^{n+r+1} = 0$$

Make exponents match: $k + r - 1$

(I avoid negative starting indices)

$$\sum_{k=0}^{\infty}(k+r)(2k+2r-2+5)c_k x^{k+r-1} + \sum_{k=2}^{\infty} c_{k-2}x^{k+r-1} = 0$$

Make indices match:

$$\underbrace{r(2r+3)c_0x^{r-1}}_{k=0} \overset{=0}{} + \underbrace{(1+r)(2r+5)c_1x^{r}}_{k=1} \overset{=0}{} + \sum_{k=2}^{\infty}\left[(k+r)(2k+2r+3)c_k + c_{k-2}\right]x^{k+r-1} = 0$$

The term with the smallest ~~subscript~~ exponent gives us r:

Here, $r(2r+3)=0 \Rightarrow r=0, -\frac{3}{2}$

It follows that $c_1=0$.

Recurrence relation: $c_k = -\frac{1}{(k+r)(2k+2r+3)}c_{k-2}$, $k=2,3,4,\ldots$

	$r=0$	$r=-\frac{3}{2}$
	$c_k = -\frac{1}{k(2k+3)}c_{k-2}$ $k=2,3,4,\ldots$	$c_k = -\frac{1}{(k-3/2)(2k)} = -\frac{1}{k(2k-3)}c_{k-2}$ $k=2,3,4,\ldots$
$k=2$	$c_2 = -\frac{1}{14}c_0$	$c_2 = -\frac{1}{2}c_0$
$k=3$	$c_3 = (\quad)c_1 = 0$	$c_3 = (\quad)c_1 = 0$
$k=4$	$c_4 = -\frac{1}{44}c_2 = \frac{1}{616}c_0$	$c_4 = -\frac{1}{20}c_2 = \frac{1}{40}c_0$

Relabel: this c_0 becomes C_1; this c_0 becomes C_2

$$y = C_1\Big(1 - \frac{1}{14}x^2 + \frac{1}{616}x^4 + \cdots\Big) + C_2x^{-3/2}\Big(1 - \frac{1}{2}x^2 + \frac{1}{40}x^4 + \cdots\Big)$$

x^0

This is the Frobenius thing

Smoother presentation regarding the indicial equation: $x^n y'' + a_1(x) y' + a_0(x) y = 0$, $n = 1$ or 2

$y'' + P(x) y' + Q(x) y = 0$ (std form)

$x^2 y'' + x^2 P(x) y' + x^2 Q(x) y = 0$ (clear any potential denoms)

Cosmetic change: $x^2 y'' + \boxed{x p(x)} y' + \boxed{q(x)} y = 0$

$p(x)$, $q(x)$ analytic at $x = 0$ means

$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots$, $q(x) = b_0 + b_1 x + b_2 x^2 + \cdots$

So, starting with $2xy'' + 5y' + xy = 0$

$$x^2 y'' + \boxed{\frac{5}{2} x} y' + \boxed{\frac{1}{2} x^2} y = 0$$

$p(x) = \frac{5}{2}$, $q(x) = \frac{1}{2} x^2$ so $a_0 = \frac{5}{2}$, $q(x) = 0$

Then $r(r-1) + a_0 r + b_0 = 0 \Rightarrow r(r-1) + \frac{5}{2} r = 0$

$\Rightarrow 2r^2 - 2r + 5r = 0 \Rightarrow 2r^2 + 3r = 0$

$r = 0, -3/2$

The equation $r(2r + 3)c_0 = 0$ is an indicial equation and can be generalized as follows:

~~$x^n y'' + a_1(x) y' + a_0(x) y = 0$, $n = 1$ or 2~~

~~$y'' + \frac{a_1(x)}{x^n} y' + \frac{a_0(x)}{x^n} y = 0$~~

~~$x^2 y'' + \frac{a_1(x)}{x^{n-2}} y' + \frac{a_0(x)}{x^{n-2}} y = 0$~~

~~Has the form $x^2 y'' + x p(x) y' + q(x) y = 0$~~

~~and $p(x) = x P(x)$, $q = x^2 Q(x)$~~

using $p(x) = xP(x) = \underline{a_0 + a_1x + a_2x^2 + \cdots}$

and $q(x) = x^2Q(x) = b_0 + b_1x + b_2x^2 + \cdots$

$$x^2y'' + x\boxed{[xP(x)]}y' + \boxed{[x^2Q(x)]}y = 0 \quad (13)$$

is $x^2y'' + x\boxed{p(x)}y' + \boxed{q(x)}y$

For $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ and approiate y', y'',

(13) becomes

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r} + \underline{(a_0 + a_1x + a_2x^2 + \cdots)}\sum_{n=0}^{\infty}(n+r)c_nx^{n+r}$$

$$+ \underline{(b_0 + b_1x + b_2x^2 + \cdots)}\sum_{n=0}^{\infty} c_nx^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\Big[(n+r)(n+r-1) + (a_0 + a_1x + a_2x^2 + \cdots)(n+r)$$
$$+ (b_0 + b_1x + b_2x^2 + \cdots)\Big]c_nx^{n+r} = 0$$

For $n=0$ $[r(r-1) + a_0r + b_0]c_0x^r = 0$

(the term with the smallest exponent on x)

$\boxed{r(r-1) + a_0r + b_0 = 0}$, where

a_0 is the constant of $p(x)$ and

b_0 is the constant of $q(x)$.

This is the indicial equation.

~~$x^2y'' + x^2P(x)y' + x^2Q(x)y = 0$~~

~~For $2xy'' + 5y' + xy = 0$ multiply by $\frac{x}{2}$~~

~~$x^2y'' + \frac{5}{2}xy' + \frac{1}{2}x^2y = 0$~~

~~is $x^2y'' + xp(x)y' + q(x)y = 0$~~

Example: Use the method of Frobenius to obtain two linearly independent solutions about the regular singular point $x = 0$.

$$9x^2y'' + 9x^2y' + 2y = 0$$

1st: Find indicial roots

$$x^2y'' + \boxed{x^2}y' + \boxed{\frac{2}{9}}y = 0$$

$xp(x) = x^2 \Rightarrow p(x) = x + 0 \Rightarrow a_0 = 0$ $\qquad$ $q(x) = \frac{2}{9} = b_0$

$$r(r-1) + a_0r + b_0 = 0 \Rightarrow r^2 - r + \frac{2}{9} = 0$$

$$9r^2 - 9r + 2 = 0 \Rightarrow (3r-1)(3r-2) = 0$$

$r = \frac{1}{3}, \frac{2}{3}$ $\quad$ *Note: These don't differ by an integer*

Solve: $$\sum_{n=0}^{\infty} 9(n+r)(n+r-1)c_n x^{n+r} + \sum_{n=0}^{\infty} 9(n+r)c_n x^{n+r+1} + \sum_{n=0}^{\infty} 2c_n x^{n+r} = 0$$

Exponents: $(k+r)$

$$\sum_{k=0}^{\infty}\left[9(k+r)(k+r-1)+2\right]c_k x^{k+r} + \sum_{k=1}^{\infty} 9(k+r-1)c_{k-1}x^{k+r} = 0$$

Indices: $k=0$: $[9r(r-1)+2]c_0x^r=0$

(indicial equation from term w/ smallest exponent)

$$9r^2-9r+2=0$$

$$\Rightarrow r=\frac{1}{3},\ \frac{2}{3}$$ (as we found above)

Combine sigmas:

$$\sum_{k=1}^{\infty}\left\{\underline{[9(k+r)(k+r-1)+2]c_k}+\underline{9(k+r-1)c_{k-1}}\right\}x^{k+r}=0$$

Recurrence relation: $c_k=-\dfrac{9(k+r-1)}{9(k+r)(k+r-1)+2}c_{k-1},\ k=1,2,3\ldots$

with $r=\frac{1}{3},\ \frac{2}{3}$

$$r=\frac{1}{3}\qquad\qquad r=\frac{2}{3}$$

$$c_k=-\frac{9(k-\frac{2}{3})}{9(k+\frac{1}{3})(k-\frac{2}{3})+2}c_{k-1}\qquad c_k=-\frac{9(k-\frac{1}{3})}{9(k+\frac{2}{3})(k-\frac{1}{3})+2}c_{k-1}$$

$$c_k=-\frac{3(3k-2)}{(3k+1)(3k-2)+2}c_{k-1}\qquad c_k=-\frac{3(3k-1)}{(3k+2)(3k-1)+2}c_{k-1}$$

$$c_k=-\frac{3(3k-2)}{9k^2-3k}c_{k-1}\qquad c_k=-\frac{3(3k-1)}{9k^2+3k}c_{k-1}$$

$$3k(3k-1)\qquad\qquad 3k(3k+1)$$

For $k=1,2,3,\ldots$

$$c_k=-\frac{3k-2}{k(3k-1)}c_{k-1}\qquad c_k=-\frac{3k-1}{k(3k+1)}c_{k-1}$$

$k=1$ $\quad c_1=-\dfrac{1}{2}c_0 \qquad c_1=-\dfrac{2}{4}c_0=-\dfrac{1}{2}c_0$

$k=2$ $\quad c_2=-\dfrac{4}{2(5)}c_1=\dfrac{1}{5}c_0 \qquad c_2=-\dfrac{5}{14}c_1=\dfrac{5}{28}c_0$

Sub: $C_1=c_0$ $\qquad C_2=c_0$

$$y = C_1 x^{1/3}\left(1 - \frac{1}{2}x + \frac{1}{5}x^2 + \cdots\right) + C_2 x^{2/3}\left(1 - \frac{1}{2}x + \frac{5}{28}x^2 + \cdots\right)$$

2 solutions:

$$y_1 = x^{1/3} - \frac{1}{2}x^{4/3} + \frac{1}{5}x^{7/3} + \cdots, \quad y_2 = x^{2/3} - \frac{1}{2}x^{5/3} + \frac{5}{28}x^{8/3} + \cdots$$

If r_1 and r_2 differ by an integer–that is, if $r_2 - r_1 \in \mathbb{Z}$–we can use r_1 to find y_1 and then reduction of order to find r_2.

Example: $x = 0$ is a regular singular point of the given differential equation. Show that the indicial roots of the singularity differ by an integer. Use the method of Frobenius to obtain at least one series solution about $x = 0$. Use reduction of order and a CAS to find a second solution. Form the general solution on $(0, \infty)$.

$$y'' + \frac{3}{x}y' - 2y = 0$$

Form: $x^2y'' + xp(x)y' + q(x)y = 0$

$$x^2y'' + 3xy' - 2x^2y = 0 \Rightarrow a_0 = 3, \; b_0 = 0$$

$$r(r-1) + 3r = 0 \Rightarrow r^2 + 2r = 0 \Rightarrow r = 0, -2$$

* The indicial roots differ by an integer *

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} + \sum_{n=0}^{\infty} 3(n+r)c_n x^{n+r-2} - \sum_{n=0}^{\infty} 2c_n x^{n+r} = 0$$

Exponents: $(k+r-2)$

$$\sum_{k=0}^{\infty} \overset{(k+r)(k+r+2)}{\cancel{(k+r)(k+r-1+3)}}\, c_k x^{k+r-2} - \sum_{k=2}^{\infty} 2c_{k-2}x^{k+r-2} = 0$$

$$k=0: \; r(r+2)c_0 x^{r-2} = 0 \Rightarrow r = 0, -2$$

$$k=1: \; (r+1)(r+3)c_1 x^{r-1} = 0 \Rightarrow c_1 = 0$$

$$\sum_{k=2}^{\infty}\left[(k+r)(k+r+2)c_k - 2c_{k-2}\right]x^{k+r-2} = 0$$

$c_k = \frac{2}{(k+r)(k+r+2)} c_{k-2}$, $k = 2, 3, 4, \dots$ and $r = 0, -2$

$r = 0$

$c_k = \frac{2}{k(k+2)} c_{k-2}$

for $k = 2, 3, 4, \dots$

~~$r = -2$~~

~~$c_k = \frac{2}{k(k-2)} c_{k-2}$~~

~~undefined if $k = 2$~~

$k = 2 \quad c_2 = \frac{1}{4} c_0$

$k = 3 \quad c_3 = \frac{2}{15} c_1 = 0$

$k = 4 \quad c_4 = \frac{1}{12} c_2 = \frac{1}{48} c_0$

It turns out that

$$y_1 = 1 + \frac{1}{4}x^2 + \frac{1}{48}x^4 + \frac{1}{1152}x^6 + \cdots$$

Recall reduction of order:

$$y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx$$

$P(x) = \frac{3}{x}$ from original equation $\Rightarrow e^{-\int P\,dx} = x^{-3}$

$$y_2 = y_1 \int \frac{1}{x^3 \; y_1^2}\,dx$$

$$y_2 = y_1 \left(\frac{7}{96}x^2 - \frac{1}{2x^2} - \frac{1}{2}\ln 2 + \cdots \right)$$

General solution: $y = c_1 y_1 + c_2 y_2$